Poset Combinatorics and Permutations

Jennifer Elder $<$ Jeremy Martin, Bruce Sagan, and Richard Stanley

Contents

1 Posets

Partially ordered sets, known as "posets" for short, give a fruitful way of ordering combinatorial objects. In this way they provide new perspectives on objects that you may already be familiar with. We will discuss poset properties, operations on posets, and then finish the section with how we wish to tie posets into the study of parking functions.

1.1 Introduction to Posets

Definition 1.1. A **partially ordered set** or **poset** is a pair $(P, \leq)$, where P is a set equipped with a relation $\leq$, that is reflexive, antisymmetric, and transitive. That is, for all $x, y \in P$:

1. Reflexivity: $x \leq x$
2. Antisymmetry: if $x \leq y$ and $y \leq x$, then $x = y$
3. Transitivity: If $x \leq y$ and $y \leq z$ then $x \leq z$

We will say that $x, y \in P$ are **comparable** if $x \leq y$ or $y \leq x$. Otherwise they are incomparable. A poset where every pair of elements is comparable is called a **total order**. We say that x is **covered** by y, written as $x \lessdot y$, if $x < y$ and there exists no z such that $x < z < y$.

Example 1.1. For each example below, determine whether the poset is a total order, and what the cover relations look like:

1. **The chain of length n** is $(C_n, \leq)$, where $C_n = \{0, 1, \ldots n\}$, and $i \leq j$ is the usual ordering of integers.

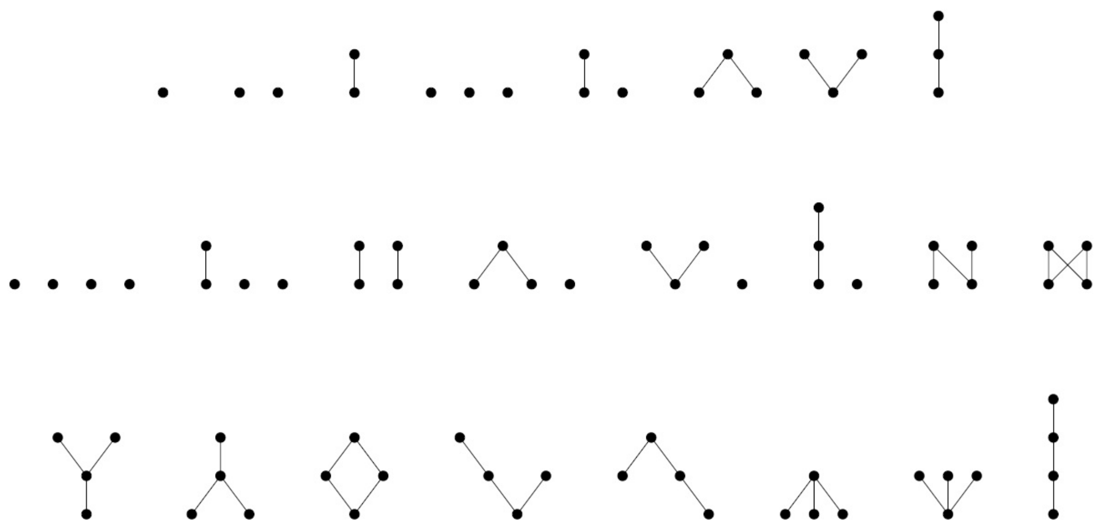
2. **The Boolean algebra** is $(B_n, \subseteq)$ where B_n is the set of all subsets of $[n] = \{1, \ldots n\}$, where $S \subset T$ is set containment.

3. **The divisor poset** is $(D_n, |)$, which consists of all positive integers which divide evenly into n and $a|b$ means that a divides evenly into b.

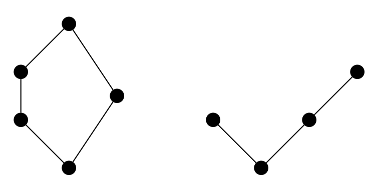
Definition 1.2. Two posets P, Q are **isomorphic** if there is a bijection $\phi : P \to Q$ that is order preserving; that is, $x \leq y$ in P iff $\phi(x) \leq \phi(y)$ in Q.

Definition 1.3. A **Hasse diagram** is a graph whose vertices are the elements of the poset and whose edges represent the cover relations, which are enough to generate all the relations in the poset by transitivity. By convention, the "largest" elements are at the top of the picture.

Below, we have the Hasse diagrams for all posets with at most four elements:



Example 1.2. Consider the following Hasse diagrams that do not appear to be present above:



Can you define an isomorphism from these diagrams to any of the Hasse diagrams above? Why or why not?

Problem 1.1. For each example below, draw the Hasse diagram of the poset.

1. **The chain of length 4** is $(C_n, \leq)$, where $C_n = \{0, 1, \ldots n\}$, and $i \leq j$ is the usual ordering of integers.

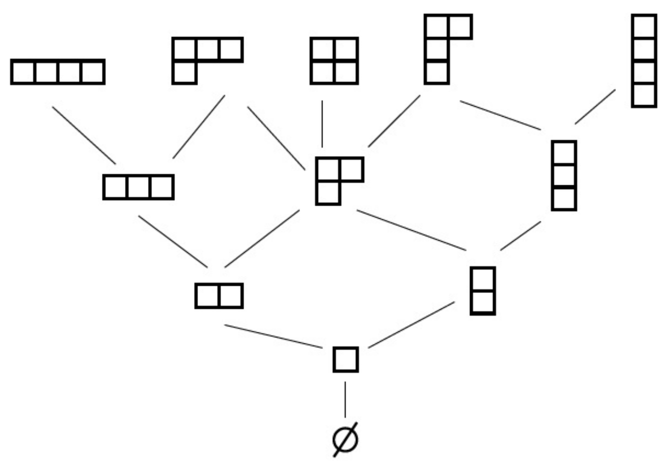
2. **The Boolean algebra** B_n for $n = 2, 3$.

3. **The divisor posets** D_6 and D_{12}

Finally, we will look at one more example.

Definition 1.4. Let $n \geq 0$ be an integer. An **integer partition of n into k parts**, written as $\lambda \vdash n$, is a multiset $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$ and $\lambda_1 + \ldots + \lambda_k = n$.
The integer partitions of n can be represented visually as **Young diagrams**, as seen below.

Example 1.3. Young's Lattice (which is a special class of poset) is an infinite poset. A portion of this poset is seen below:



The cover relations are defined as full containment of one diagram of n boxes inside a diagram of $n+1$ boxes.

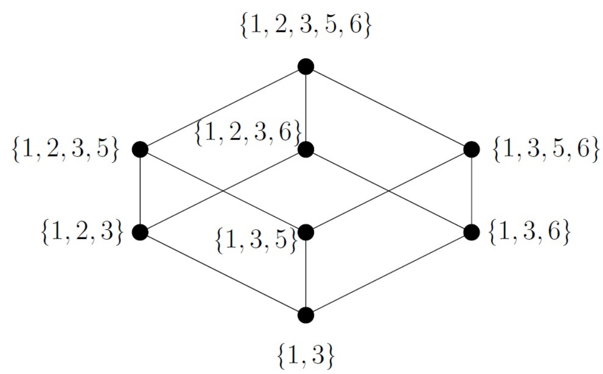
1.2 Structures and Operations on Posets

Definition 1.5. A **weak subposet** of a poset P is a subset $Q \subseteq P$ such that if $x \leq_Q y$ in Q, then $x \leq_P y$ in P. If $Q = P$ as sets, but Q is not equal to P as a poset, then we call Q a **refinement** of P. An **induced subposet** is a subset Q such that $x \leq_Q y$ if and only if $x \leq_P y$.

Problem 1.2. Find at least three non-trivial subposets for D_{12}.

Example 1.4. Consider the following subposet of B_6:

$[\{1,3\},\ \{1,2,3,5,6\}] =$



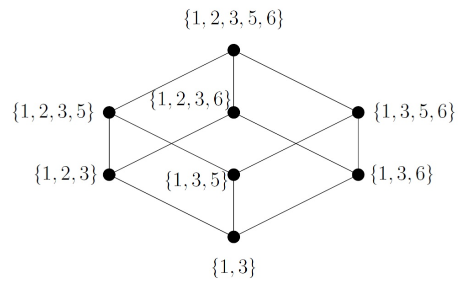
Is this an induced subposet? Why or why not?

Definition 1.6. An **interval** of a poset P denoted $[x, y] = \{u \in P \mid x \leq u \leq y\}$ is an induced subposet of P that is defined whenever $x \leq_P y$.

The Boolean subposet above is an interval of B_6.

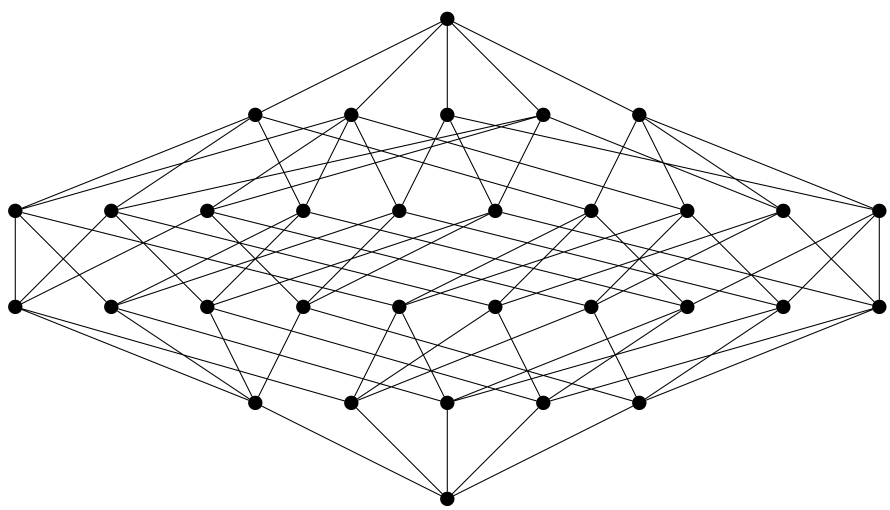
Definition 1.7. A **chain** C is a poset in which any two elements are comparable. The **length** of the chain, denoted $\ell(C)$ is defined as one less than the number of elements in the chain. A **maximal chain** is a chain in P that is not properly contained in any longer chains in P.

Example 1.5. Let's look at the subposet of B_6 again:

$[\{1,3\},\ \{1,2,3,5,6\}] =$ 

We want to find chains of length 0, 1, 2, and 3, and discuss whether these chains are maximal or not.

Problem 1.3. Find all maximal chains in B_2, B_3, B_4, and B_5 (seen below). Use your work to conjecture how many maximal chains will exist in B_n.



Definition 1.8. Let $n \in \mathbb{N}$. A **set partition of** $[n]$, denoted Π_n, is a collection of disjoint subsets of the set $[n]$ whose union is the full set. A **block** is one of these subsets.
We can make the set Π_n of all **partitions of** $[n]$ into a poset by defining $x \leq y$ in Π_n if every block of x is contained in a block of y.
For example, if x has blocks 137, 2, 46, 58, and 9, and y has blocks 13467 and 2589, then $x \leq y$. We say that x is a **refinement** of y.
A **noncrossing set partition** is a partition of $[n]$ such that for elements $a < b < c < d$, we do not have ac in one black and bd in another block. The example above is not a noncrossing partition, but 13, 2, 4 is a noncrossing partition of $[4]$.

Example 1.6. Let's consider the poset Π_3.

Problem 1.4. What if we only care about the non-crossing partitions? Find Π_4, and the subposet that only contains non-crossing partitions

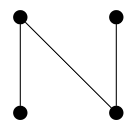
Definition 1.9. Let P and Q be two posets. Then the **direct product** of P and Q is the poset

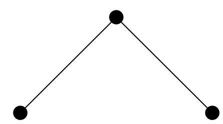
$$P \times Q = \{(s,t) \mid s \in P, t \in Q\}$$

where $(x,y) \leq_{P\times Q} (x',y')$ if $x \leq_P x'$ and $y \leq_Q y'$.
We will use P^n to denote the product $P \times P \times \cdots \times P$ where P is present n times.

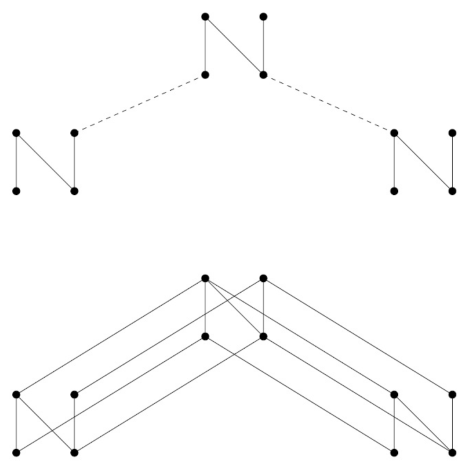
In order to draw the Hasse diagram of $P \times Q$, draw a copy of P, replace each element $x \in P$ with a copy of Q, and connect the corresponding elements of Q_x and Q_y when $x \lessdot_P y$.

Example 1.7. Consider the following posets, P and Q:





Below is the image for $P \times Q$



Do you think $Q \times P$ would look different? Would the posets actually be different?

It was easy to see that the poset we arrived at was the direct product of P and Q. For some problems, it is useful to start with a poset and try to **decompose** it as a direct product of other posets.

Example 1.8. Let's look at C_1, $C_1 \times C_1$, and $C_1 \times C_1 \times C_1$ and see if we recognize anything.

Exercise Set - Posets

1. Show that there is an isomorphism between $P \times Q$ and $Q \times P$.
2. Find a Boolean poset B_n that is isomorphic to the interval in Example 1.4 (you may already have it in your notes). Define the isomorphism between these posets.
3. Consider D_{12} again. Use the fact that $12 = 2^2 \cdot 3^1$ and see if you can decompose this poset into a product of chains. What do you conclude about general D_n?
4. Consider the following integer partitions of n: $1 + 1 + \ldots + 1$ and n. Define bijections between certain chains in the Young Lattice and C_n using each specific type of integer partition. What will these isomorphisms look like?
5. How do you find the number of maximal chains in Π_n?

Notes

2 Permutations

Permutations are one of the most interesting and useful objects in combinatorics. Part of why permutations are so useful is the flexibility they offer. We can count properties of permutations using combinatorial proofs. We can find bijections from sets we want to study to sets of permutations, and use the permutations to answer whatever questions we may have. In this section, we will look at different permutation properties, proofs that use properties of permutations, and pattern avoidance. We will wrap up the section by discussing how we want to use permutations to study functions.

2.1 Introduction to Permutations

You may or may not have seen permutations in your math classes before now. Either way, we will have a quick refresher on the basics.

Definition 2.1. Let A be a finite set containing n elements. We define S_A as the set of bijections from A to itself. For simplicity's sake, we write this as $A = \{1, 2, \ldots, n\}$, or $A = [n]$, and use the notation S_n. This is a group under composition of functions, and the elements in this set are called permutations.

There are many different ways to write permutations. We will go over a few of those in the next example:

Example 2.1. We can consider a permutation as a **two line array**, where the image of element $j \in [n]$ appears directly below it. For example,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 6 & 5 & 1 \end{pmatrix}$$

The first line in the array is always the same, 1 up to n. We can write the same permutation in **one line notation**:

$$\sigma = 432651$$

We can also look at this function in **cycle notation**,

$$\sigma = (1\ 6)(2\ 3)(1\ 4)(5)$$

and as a product of **disjoint cycles**

$$\sigma = (1\ 4\ 6)(2\ 3)(5)$$

In general, we will write functions in S_n using Greek letters like σ, and we will write the one line notation for such a permutation as $\sigma = \sigma_1 \sigma_2 \ldots \sigma_n$.

Problem 2.1. If the permutations below are written in one line notation, write the permutation as a product of disjoint cycles. If the permutation is written as a product of cycles, write the one line form of the function, and specify which S_n you want to consider for the permutation.

1. $\alpha = 5\ 4\ 3\ 2\ 1$
2. $\beta = (1\ 3)(2\ 4\ 6\ 5)$
3. $\gamma = (1\ 2)(2\ 3)(4\ 5)(1\ 2)(5\ 6)$

Definition 2.2. Let $\sigma \in S_n$. Then there exists a permutation $\tau \in S_n$ such that $\tau\sigma = Id = \sigma\tau$, and we will denote this permutation as σ^{-1}.

Example 2.2. When it comes to finding the inverse of σ, it can be easier to work with cycle notation. Consider for example, $\sigma = (1\ 4\ 6)(2\ 3)(5)$

For the next two definitions, the one line notation for the permutation is more useful than the cycles.

Definition 2.3. For $\sigma \in S_n$, written in one line notation as $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$, we define the **descent set of** σ as follows: $D(\sigma) = \{i \in [n] \mid \sigma_i > \sigma_{i+1}\}$.

Descents are easy to read off of the one line version of the permutation, no matter what n is. The next definition takes a bit more work for large values of n:

Definition 2.4. For $\sigma \in S_n$, written in one line notation as $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$, we define an **inversion** (sometimes **left inversions**) of σ as a pair (σ_i, σ_j) where $i < j$ and $\sigma_i > \sigma_j$. The **inversion number** of σ, denoted inv(σ) is the size of the set containing all such inversion pairs.

Example 2.3. Consider $\sigma = 432651$. We want to find $D(\sigma)$ and inv(σ).

Definition 2.5. The **major index** of σ, denoted maj(σ) is a permutation statistic defined as

$$\text{maj}(\sigma) = \sum_{i \in D(\sigma)} i$$

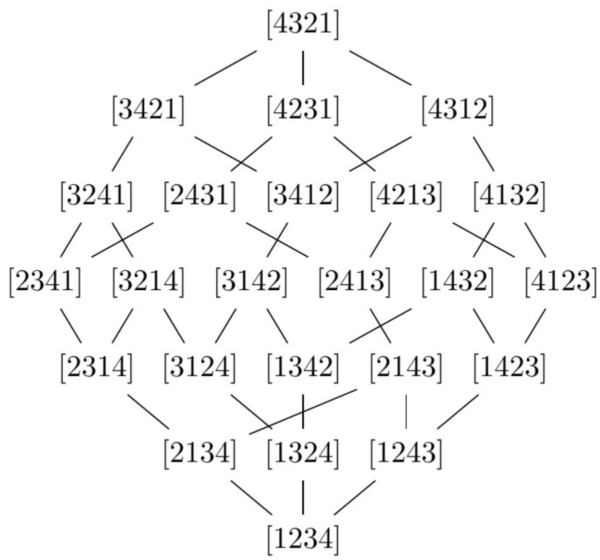
Definition 2.6. The **Lehmer code** of σ, denoted $L(\sigma)$ is an integer sequence $(L(\sigma_1), L(\sigma_2), \ldots L(\sigma_n))$, where $L(\sigma_i) = |\{j > i \mid \sigma_i > \sigma_j\}|$. Each number in the code is bounded: $0 \leq L(\sigma_i) \leq n - i$.

Example 2.4. Consider $\sigma = 432651$. We want to find the major index and the Lehmer code for this permutation.

Inversions also allow us to define a partial order for S_n.

Definition 2.7. The **right weak order poset** is a partial order defined on S_n, denoted $W(S_n)$. This partial order is a lattice for all $n \in \mathbb{N}$
The cover relations are defined on the addition or removal of a single inversion pair.

Below is the lattice for S_4:



There are many more statistics on permutations that we have not discussed here. For more information, you can go to

```
http://www.findstat.org/
```

and explore the database of existing statistics. FindStat also includes Sage code for each statistic as part of the database entry.

If you want a list of commands in one place, you can go to

```
https://doc.sagemath.org/html/en/reference/combinat/sage/combinat/permutation.html
```

For example, to find all the inversion pairs for $\sigma = 76124385$, you can save a lot of time by running the code

```
Permutation([7,6,1,2,4,3,8,5]).inversions()
```

in Sage.

2.2 Fun with Permutations

Let R be an integral domain. We will consider the polynomial ring $R[x_1, \ldots, x_n]$, and define a group action on $R[x_1, \ldots, x_n]$ as follows: let $\sigma \in S_n$ and $f(x_1, x_2, \ldots x_n) \in R[x_1, \ldots, x_n]$. Then we let

$$\sigma f(x_1, x_2, \ldots x_n) = f(x_{\sigma_1}, x_{\sigma_2}, \ldots x_{\sigma_n})$$

We are often interested in **invariance** when we study permutations. This leads us to the following definition:

Definition 2.8. A **symmetric polynomial** is a polynomial in $R[x_1, \ldots, x_n]$ that is invariant under permutation of the variables. We consider these polynomials up to scalar multiplication and linear combinations.
The degree of a monomial $m = x_{b_1}^{a_1} x_{b_2}^{a_2} \ldots x_{b_l}^{a_l}$ has degree $deg(m) = \sum_i a_i$. We say that $f \in R[x_1, \ldots, x_n]$ is **homogeneous of degree** d if each monomial in f is of degree d. Sometimes we will simply refer to this as f is of degree d.

Example 2.5. Let $n = 3$, and consider the variables x, y, z. Then the only homogeneous symmetric polynomial of degree 1 is $x + y + z$. Why is this statement true?

Problem 2.2. Let $n = 3$, and consider the variables x, y, z. Find the only symmetric polynomials of degree 2, up to scalar multiplication.

Definition 2.9. The number $c(n,k)$ is defined as the number of permutations in S_n with exactly k cycles.

Lemma 2.1. The numbers $c(n,k)$ satisfy the recurrence

$$c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1)$$

for $n, k \geq 1$, with the initial conditions $c(n,k) = 0$ if $n < k$ or $k = 0$, except $c(0,0) = 1$

Problem 2.3. Before we discuss the general proof, let's try to figure out what is going on here. Let $k = 2$ and $n = 4$. We want to use the number of permutations in S_3 with one or two cycles in order to count the number of permutations in S_4 with exactly two cycles. How do you take a permutation in S_3 and make it into a permutation in S_4? Note: $c(4,2) = 11$.

Definition 2.10. Let x be a variable. A sequence $(a_0, a_1, a_2 \ldots, a_n)$ of complex numbers has **ordinary generating polynomial**

$$f(x) = a_0 + a_1 x^1 + \ldots + a_n x^n = \sum_{k=0}^{n} a_k x^k$$

Theorem 2.1. For $n \geq 2$, the generating function for the number of permutations σ with $inv(\sigma) = k$, denoted a_k can be written as

$$\sum_{k=0}^{n} a_k q^k = \sum_{\sigma \in S_n} q^{\mathsf{inv}(\sigma)}$$

and furthermore, this generating polynomial will factor as

$$\sum_{\sigma \in S_n} q^{\mathsf{inv}(\sigma)} = (1+q)(1+q+q^2)\ldots(1+q+\ldots+q^{n-1})$$

In your exercise set, you will show that the Lehmer code is such that $\mathsf{inv}(\sigma) = \sum_{i \in [n]} L(\sigma_i)$. We will use that fact here to prove the theorem.

Definition 2.11. Let λ be an integer partition of n (written $\lambda \vdash n$) into k parts, such that the sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ is non-decreasing and $\lambda_1 + \ldots + \lambda_k = n$. A **standard Young tableau of shape λ** is a bijective filling of the Young diagram of shape λ with the elements of $[n]$ so that each row and column is weakly increasing. The number of standard Young tableaux of shape λ is denoted f^λ

Example 2.6. Below is a standard Young tableau of shape $\lambda = (3, 3, 1)$.

1	2	5
3	6	7
4		

Find the two standard Young tableau of shape $\lambda = (2, 1)$:

Theorem 2.2. For any given $n \geq 0$, we have

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$$

The key to the proof of this theorem is constructing a bijection between S_n and pairs of standard young tableau. First we need the bumping algorithm defined below:

1. Set $R :=$ the first row of diagram P
2. While x is less than some element of row R, let y be the leftmost such element. Replace y with x in R. Repeat this step with $R :=$ the row below R, and $x := y$.
3. Now x is greater than every element of R, so place x at the end of this row and terminate.

The first tableau is constructed using the bumping algorithm that will take all the entries of $\sigma = \sigma_1 \ldots \sigma_n$ from left to right, while the second tableau will record the order in which the boxes were added to P.

Example 2.7. Consider $\sigma = 15423$.

Exercise Set - Permutations

1. For a complete answer, you will need to find the code that will answer each part of the problem, and the answer returned by Sage. Consider the permutation $\sigma = 415973268$. Answer the following questions:
   (a) Find the disjoint cycle decomposition
   (b) Find the descent set
   (c) Find the number of inversions
   (d) Find the longest increasing subsequence present in the one line notation
   (e) Find the Lehmer Code
   (f) Find the fixed points
   (g) Find the pair of standard Young tableaux given by the Robinson Schensted algorithm.
2. Show that the Lehmer code is such that $\text{inv}(\sigma) = \sum_{i\in[n]} L(\sigma_i)$.
3. Find the only homogeneous symmetric polynomials of degree 3 using x, y and z.
4. Find the pair of standard Young tableaux that correspond to the permutations $\sigma = 15432$ and $\tau = 321546$.
5. Work through a few examples, and then use the definition of an inversion pair to discuss why we must have that $\text{inv}(\sigma) = \text{inv}(\sigma^{-1})$ for any permutation σ.

Notes

3 Appendix

More on Posets

Definition 3.1. A poset L is a **lattice** if every pair $x, y \in L$ has the following:

1. a unique largest common lower bound, called their **meet**, written $x \wedge y$,
2. a unique smallest common upper bound, called their **join**, written $x \vee y$.

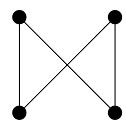
Then for all $z \in L$,

$$z \leq x \text{ and } z \leq y \implies z \leq x \wedge y$$

$$z \geq x \text{ and } z \geq y \implies z \geq x \vee y$$

Note that the diagrams you created in Problem 1.1 are all lattices.

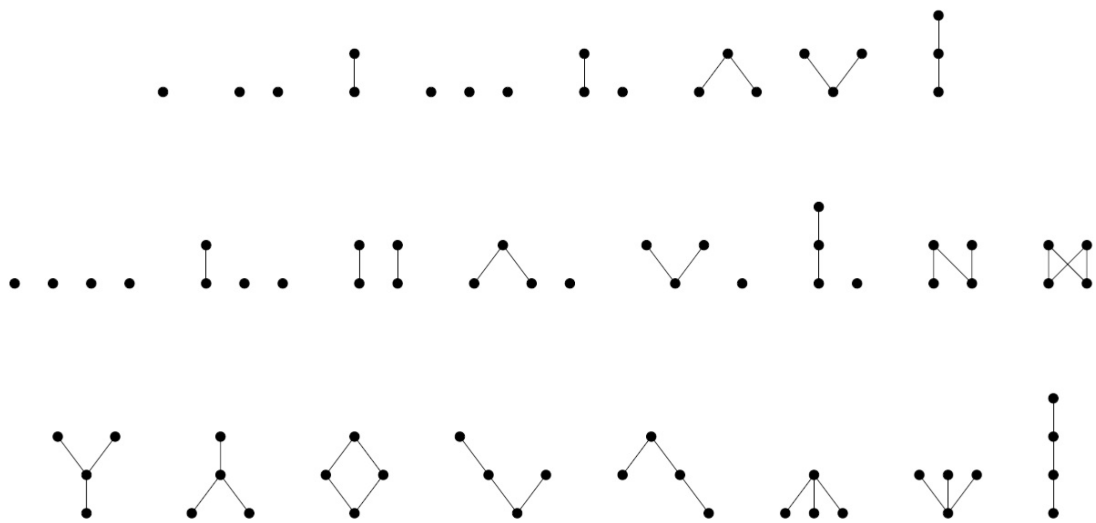
Example 3.1. For the Hasse diagram below, give two explanations for why this poset is not a lattice.



Problem 3.1. Using 5 elements, create Hasse diagrams for the following types of posets:

1. Fails condition 1 of being a lattice, while condition 2 holds
2. Fails condition 2 of being a lattice, while condition 1 holds
3. Both conditions hold, and it is a lattice

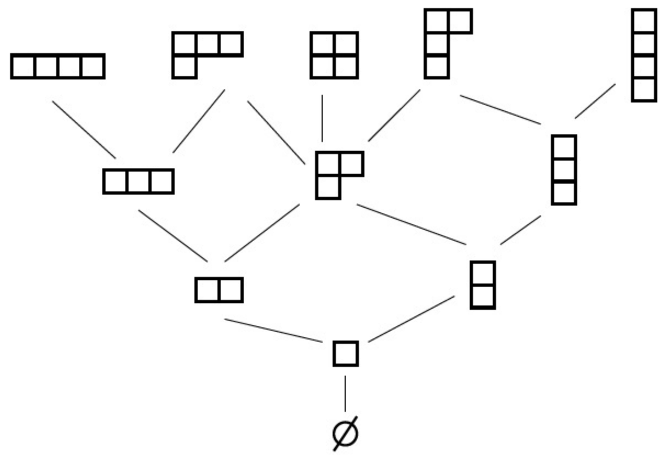
Problem 3.2. For each of the Hasse diagrams below, label whether the diagram is a lattice or not.



Definition 3.2. A **minimal element** of a poset P is an element x such that there is no $y \in P$ with $y < x$. A **maximal element** of a poset P is an element w such that there is no $y \in P$ with $y > w$. Note that minimal and maximal elements need not be unique.
If P contains a unique minimal element, then we call that element $\hat{0}$. If P has a unique maximal element, we call that element $\hat{1}$.

Example 3.2. If you look back at all the previous examples of lattices that we have discussed, what do you notice about minimal or maximal elements?

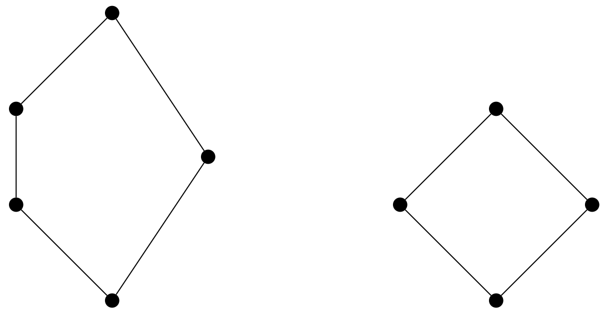
Example 3.3. We again return to Young's Lattice, which is an infinite lattice. A portion of this lattice is seen below:



The cover relations are defined as full containment of one diagram of n boxes inside a diagram of $n+1$ boxes. This is a Lattice with no $\hat{1}$, but all joins are well defined. How does this lattice match up with your original answer to Example 3.2? What do you think the join of any two diagrams will look like?

Definition 3.3. If all maximal chains in P have the same length, m, then we say that P is **graded of rank** m. In this case, there is a unique **rank function** $\rho : P \to \{0, 1, \ldots n\}$ such that $\rho(x) = 0$ if x is minimal in P, and $\rho(y) = \rho(x) + 1$ if $x \lessdot y$. In general, if $x \leq y$, then $\rho(x, y) = \rho(y) - \rho(x) = \ell(x, y)$.

Example 3.4. Which one of the posets below is not ranked?



Example 3.5. Consider the poset D_{12}. We want to be able to describe the rank of an element, and the rank of the poset.

What would you say in general about the rank of elements in D_n? What about the rank of the poset?

Problem 3.3. What is the rank of an element in the Young Lattice? What about the Möbius function of an element? For either of these functions, can you write your answer in terms of the number of boxes in the Young Diagram?

Definition 3.4. The **Möbius Function** for elements in a poset P, denoted $\mu(x)$, is defined as follows:

$$\mu(x) = 1 \text{ if } x = \hat{0}$$

$$\mu(x) = -\sum_{y<x} \mu(y)$$

Example 3.6. Let's compute $\mu(x)$ for the elements in C_3 and D_3

Problem 3.4. What do you think will be true for $\mu(x)$ in C_n? What about $\mu(x)$ in D_n?

Definition 3.5. The **Möbius Function** for intervals of a poset P, denoted $\mu(x, y)$, is defined as follows:

$$\mu(x, y) = 1 \text{ if } x = y$$

$$\mu(x, y) = 0 \text{ if } x > y$$

$$\mu(x, y) = -\sum_{x \leq t < y} \mu(x, t)$$

If P has minimum and maximum elements $\hat{0}$ and $\hat{1}$, then we can simplify the notation to

$$\mu(P) = \mu(\hat{0}, \hat{1})$$

Example 3.7. Let's compute $\mu(P)$ for the posets C_3 and D_3.

One of the nicer results for the Möbius function is that for direct products, $P \times Q$, then

$$\mu(P \times Q) = \mu(P)\mu(Q)$$

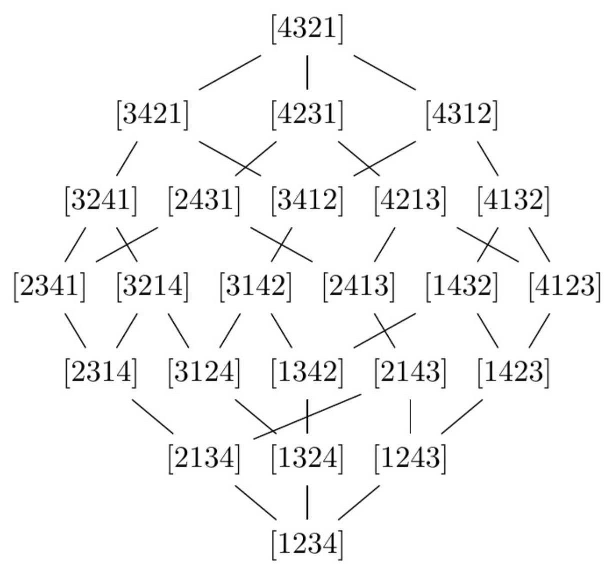
which can naturally be extended to larger products. Use this property to find $\mu(C_n)$, $\mu(B_n)$ and $\mu(D_n)$.

Pattern Avoidance

Definition 3.6. Let $\sigma = [a_1 \dots a_n]$, and $p = p_1 \dots p_m$, for $m \leq n$ and $p_i \in \mathbb{Z}_{>0}$ for all $1 \leq i \leq m$. The permutation σ **contains the pattern p** if there exist indices $i_1 < \dots < i_m$ such that $a_{i_1} \dots a_{i_m}$ are in the same order as $p_1 \dots p_m$. That is, $p_j < p_k$ if and only if $a_{i_j} < a_{i_k}$. If σ does not contain p, then σ is **p pattern avoiding**.

For example, $\sigma = [321]$ is 312 pattern avoiding, while the permutation $\omega = [53421]$ contains the pattern 312.

Example 3.8. Let's look at S_4 again:



Let's consider the pattern 132. Using the numbers 1-4, in how many ways can we write this pattern?

Now let's sort S_4 into permutations that contain the pattern, and those that do not.

After you've seen the Catalan Numbers, you may want to come back to this section to look at this interesting bijection.

Definition 3.7. For $\sigma \in S_n$, we define the $n \times n$ matrix A_σ such that

$$(A_\sigma)_{ij} = \begin{cases} 1 & \text{if } \sigma_i = j \\ 0 & \text{else} \end{cases}$$

Example 3.9. Let's look at the matrices associated with the permutations in S_2 and S_3. What properties do you notice about these matrices?

We can take these matrices and construct Lattice Paths.

Example 3.10. Let's look at the lattice paths associated with the permutations in S_2 and S_3.

Do you recognize any of these lattice paths? What do you notice about the path associated to the permutation $\sigma = 321$?

Problem 3.5. Using a different rule to take a permutation matrix and draw a lattice path, discuss how we could find a bijection between the 132 avoiding permutations and the Catalan numbers.

Problem 3.6. Consider the sequences $(4, 1, 2, 5, 7, 3, 9, 6, 8)$ and $(4, 1, 5, 9, 7, 3, 2, 6, 8)$. Look for the patterns 321 and 132. Do these sequences contain or avoid any instances of these patterns?
Once you've tried this by hand, can you find the Sage code that will allow you to easily answers these questions?

Problem 3.7. Construct sequences of length 5 that are not strictly increasing such that

1. avoid 321
2. avoid 132
3. contains both 321 and 132

Discuss what techniques you used when constructing these sequences. What did you pay attention to?

References

[1] Jeremy Martin, Lecture Notes on Algebraic Combinatorics, (2018).

[2] Bruce Sagan, *Combinatorics: The Art of Counting* (2020).

[3] Stanley, R.P., *Enumerative Combinatorics, Vol. 1* (2nd Ed.) (2011).